Journal of Pure and Applied Algebra 214 (2010) 269–280

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

An algebraic formula for the intersection number of a polynomial immersion

Iwona Karolkiewicz, Aleksandra Nowel*, Zbigniew Szafraniec

Institute of Mathematics, University of Gdańsk, 80-952 Gdańsk, Wita Stwosza 57, Poland

ARTICLE INFO

Article history:
Received 28 November 2008
Received in revised form 6 April 2009
Available online 10 June 2009
Communicated by E.M. Friedlander

MSC:
14P25
57N35
32S50

ABSTRACT

An algorithm is presented for computing the topological degree for a large class of polynomial mappings. As an application there is given an effective algebraic formula for the intersection number of a polynomial immersion $M \to \mathbb{R}^{2m}$, where M is an m-dimensional algebraic manifold.

Let $H = (h_1, \ldots, h_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous mapping, and let p be isolated in $H^{-1}(0)$. Then one may define the local topological degree $\deg_p H$ as the topological degree of the mapping

$$S_r^{n-1} \ni x \longmapsto H(x)/\|H(x)\| \in S^{n-1},$$

where S_r^{n-1} is the sphere of radius $r \ll 1$ centered at p.

Suppose that H is a polynomial mapping and $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a polynomial. Put $U = \{x \in \mathbb{R}^n : \ u(x) > 0\}$. Let $J_{\mathbb{R}} \subset \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ denote the ideal generated by $h_1, \ldots, h_n$, and let $Q = \mathbb{R}[x]/J_{\mathbb{R}}$. Suppose that $\dim_{\mathbb{R}} Q < \infty$, so that $H^{-1}(0)$ is finite. According to [1], one may construct quadratic forms Φ_T and Ψ_T on Q such that

$$\sum \deg_p H = (\text{signature } \Phi_T + \text{signature } \Psi_T)/2,$$

where $p \in H^{-1}(0) \cap U$. In [1] also presented a simpler formula for

$$\sum \deg_p H \bmod 2.$$

The assumption $\dim_{\mathbb{R}} Q < \infty$ is too restrictive in some cases. For instance if $H^{-1}(0)$ is infinite then $\dim_{\mathbb{R}} Q = \infty$.

This paper is devoted to the case where there exists an ideal $I_{\mathbb{R}} \supset J_{\mathbb{R}}$ such that

$$(J_{\mathbb{R}} : I_{\mathbb{R}}) + I_{\mathbb{R}} = \mathbb{R}[x] \quad \text{and} \quad \dim_{\mathbb{R}} \mathbb{R}[x]/(J_{\mathbb{R}} : I_{\mathbb{R}}) < \infty.$$

(The quotient of ideals $J_{\mathbb{R}} : I_{\mathbb{R}}$ is defined as the set of all $f \in \mathbb{R}[x]$ with $fg \in J_{\mathbb{R}}$ for all $g \in I_{\mathbb{R}}$.)

We shall show that in that case $H^{-1}(0) \setminus V(I_{\mathbb{R}}) = V(J_{\mathbb{R}} : I_{\mathbb{R}})$ is finite, and there are effective algebraic formulae for $\sum \deg_p H \pmod 2$ and $\sum \deg_p H$ analogous to the ones proved in [1].

Let M be a compact oriented m-dimensional manifold ($m > 1$), and let $g : M \longrightarrow \mathbb{R}^{2m}$ be an immersion. Whitney in [2] introduced the intersection number $I(g)$.

* Corresponding author. Tel.: +48 58 5232059; fax: +48 58 3414914.
E-mail addresses: ikarolki@manta.univ.gda.pl (I. Karolkiewicz), Aleksandra.Nowel@math.univ.gda.pl (A. Nowel), Zbigniew.Szafraniec@math.univ.gda.pl (Z. Szafraniec).

0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2009.05.004

Assume that $M \subset \mathbb{R}^{n+m}$ is an algebraic complete intersection, and a polynomial immersion g has a finite set of self-intersections. Let $\Delta \subset \mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$ be the diagonal.

We show that one may construct a polynomial mapping $H : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2m}$ such that $\Delta \subset H^{-1}(0)$, $H^{-1}(0) \setminus \Delta$ is finite, and for even m

$$I(g) = \frac{1}{2} \sum_p \deg_p H,$$

where $p \in H^{-1}(0) \setminus \Delta$. There is also presented a similar formula for odd $m > 1$.

As $\Delta \subset H^{-1}(0)$ is infinite, we cannot apply the method presented in [1] so as to compute $\sum_p \deg_p H$. We show how to apply methods developed in this paper so as to express the intersection number $I(g)$ either as the signature of a quadratic form on $\mathcal{A} = \mathbb{R}[x]/(J_{\mathbb{R}} : I_{\mathbb{R}})$ (for even m) or in terms of $\dim_{\mathbb{R}} \mathcal{A}$ and signs of determinants of matrices of two quadratic forms (for odd $m > 1$).

The paper is organized as follows. In Section 1 we show that the complex algebra $\mathbb{C}[x]/(J_{\mathbb{C}} : I_{\mathbb{C}})$ is isomorphic to the product of some algebras associated to the ideal $J_{\mathbb{C}}$ and points in $H_{\mathbb{C}}^{-1}(0) \setminus V(I_{\mathbb{C}})$. In Section 2 we discuss relations between the ideal $J_{\mathbb{R}} : I_{\mathbb{R}}$ and its complex counterpart $J_{\mathbb{C}} : I_{\mathbb{C}}$. In Section 3 we recall some results presented in [1]. In particular we explain how to construct the bilinear forms which we use in Section 4, where we prove a formula for $\sum_p \deg_p H$ in terms of their signatures. In Section 5 we apply these results so as to give an effective method for computing the intersection number of a polynomial immersion on an algebraic manifold. Finally we present examples computed by a computer. We implemented our algorithm with the help of Singular [3]. We have also used computer programs written by Adriana Gorzelak and Magdalena Sarnowska — students of computer sciences at the University of Gdańsk — which can construct matrices of Φ_T, Ψ_T, and compute the signature of a bilinear form.

1. Quotients of ideals in $\mathbb{C}[x]$

If S is an ideal in $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ then by $V(S)$ we will denote the set of all complex zeros of S. For each $p = (p_1, \ldots, p_n) \in \mathbb{C}^n$, m_p is the maximal ideal generated by monomials $x_1 - p_1, \ldots, x_n - p_n$. If $J, I \subset \mathbb{C}[x]$ are ideals, then

$$J : I = \{f \in \mathbb{C}[x] : fg \in J \text{ for all } g \in I\}$$

is called the ideal quotient of J by I. Since $J \subset J : I$, $V(J : I) \subset V(J)$.

Suppose that $J, I \subset \mathbb{C}[x]$ are ideals such that $J \subset I$ and

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[x]}{J : I} < \infty.$$

Then $V(J : I)$ is finite. Applying [4, Theorem 7, p. 192] we get inclusions:

$$V(J) \setminus V(I) \subset \overline{V(J) \setminus V(I)} \subset V(J : I) \subset V(J),$$

so that $V(J) \setminus V(I) \subset V(J : I) \setminus V(I) \subset V(J) \setminus V(I)$. Denote $V(J) \setminus V(I) = V(J : I) \setminus V(I) = \{p_1, \ldots, p_s\}$.

Proposition 1. *There exists a positive integer k such that*

$$\frac{\mathbb{C}[x]}{J} \cong \frac{\mathbb{C}[x]}{J + I^k} \times \frac{\mathbb{C}[x]}{J + m_{p_1}^k} \times \cdots \times \frac{\mathbb{C}[x]}{J + m_{p_s}^k}.$$

In particular $J = (J + I^k) \cap (J + m_{p_1}^k) \cap \cdots \cap (J + m_{p_s}^k)$. The same holds true for any integer bigger than k.

Proof. We have $V(I) \subset V(J)$ and $V(J) \setminus V(I) = \{p_1, \ldots, p_s\}$, so

$$V(J) = \{p_1, \ldots p_s\} \cup V(I) = V(m_{p_1}) \cup \cdots \cup V(m_{p_s}) \cup V(I) = V(m_{p_1} \cap \cdots \cap m_{p_s} \cap I).$$

By the Hilbert Nullstellensatz, there exists k such that

$$(m_{p_1} \cap \cdots \cap m_{p_s} \cap I)^k \subset J. \tag{1}$$

If $i \neq j$ then $V(m_{p_i}^k + m_{p_j}^k) = V(m_{p_i}^k) \cap V(m_{p_j}^k) = \emptyset$, so,

$$\mathbb{C}[x] = m_{p_i}^k + m_{p_j}^k \subseteq (J + m_{p_i}^k) + (J + m_{p_j}^k) \subseteq \mathbb{C}[x]$$

and then

$$(J + m_{p_i}^k) + (J + m_{p_j}^k) = \mathbb{C}[x].$$

We have $V(I^k + m_{p_i}^k) = V(I) \cap \{p_i\} = \emptyset$, so that

$$\mathbb{C}[x] = I^k + m_{p_i}^k \subseteq (J + I^k) + (J + m_{p_i}^k) \subseteq \mathbb{C}[x],$$

and then

$$(J + I^k) + (J + m_{p_i}^k) = \mathbb{C}[x].$$

The Chinese Remainder Theorem implies that:

$$\frac{\mathbb{C}[x]}{(J + I^k) \cap (J + m_{p_1}^k) \cap \cdots \cap (J + m_{p_s}^k)} \cong \frac{\mathbb{C}[x]}{J + I^k} \times \frac{\mathbb{C}[x]}{J + m_{p_1}^k} \times \cdots \times \frac{\mathbb{C}[x]}{J + m_{p_s}^k}$$

and

$$(J + I^k) \cap (J + m_{p_1}^k) \cap \cdots \cap (J + m_{p_s}^k) = (J + I^k) \cdot (J + m_{p_1}^k) \cdot \cdots \cdot (J + m_{p_s}^k).$$

By (1) we get

$$\begin{aligned} J \subset (J + I^k) \cap (J + m_{p_1}^k) \cap \cdots \cap (J + m_{p_s}^k) &= (J + I^k) \cdot (J + m_{p_1}^k) \cdot \cdots \cdot (J + m_{p_s}^k) \subset J + (I^k \cdot m_{p_1}^k \cdot \cdots \cdot m_{p_s}^k) \\ &= J + (I \cdot m_{p_1} \cdot \cdots \cdot m_{p_s})^k \subset J + (I \cap m_{p_1} \cap \cdots \cap m_{p_s})^k \subset J + J, \end{aligned}$$

and then $(J + I^k) \cap (J + m_{p_1}^k) \cap \cdots \cap (J + m_{p_s}^k) = J$. So we have

$$\frac{\mathbb{C}[x]}{J} \cong \frac{\mathbb{C}[x]}{J + I^k} \times \frac{\mathbb{C}[x]}{J + m_{p_1}^k} \times \cdots \times \frac{\mathbb{C}[x]}{J + m_{p_s}^k}. \quad \square$$

Lemma 2. *Let $p \in V(J)$. For any positive integer k, f is invertible in $\mathbb{C}[x]/(J + m_p^k)$ if and only if $f(p) \neq 0$.*

Proof. ($\Rightarrow$) Since f is invertible in $\mathbb{C}[x]/(J + m_p^k)$, there exists $g \in \mathbb{C}[x]$ such that $fg \equiv 1 \bmod (J + m_p^k)$, so

$$fg = 1 + w + v,$$

where $w \in J$ and $v \in m_p^k$. Hence

$$f(p)g(p) = 1 + w(p) + v(p) = 1,$$

and so $f(p) \neq 0$.
($\Leftarrow$) f can be presented as $f = f(p) - g$, where $g \in m_p$. Then

$$(f(p) - g)(f(p)^{k-1} + f(p)^{k-2}g + f(p)^{k-3}g^2 + \cdots + g^{k-1}) = f(p)^k - g^k.$$

Set $h = (f(p)^{k-1} + f(p)^{k-2}g + \cdots + g^{k-1})$. We have $g^k \in m_p^k$, so $fh = (f(p) - g)h \equiv f(p)^k \neq 0$ in $\mathbb{C}[x]/(J + m_p^k)$, and then f is invertible in $C[x]/(J + m_p^k)$. $\square$

Take k as in Proposition 1. Then

Lemma 3. $J : I = (\bigcap_{i=1}^s (J + m_{p_i}^k)) \cap ((J + I^k) : I)$.

Proof. Take $f \in \bigcap_{i=1}^s (J + m_{p_i}^k) \cap ((J + I^k) : I)$. For any $g \in I$ and $1 \leq i \leq s$ we have

$$fg \in J + I^k \quad \text{and} \quad fg \in J + m_{p_i}^k,$$

so $fg \in (J + I^k) \cap (J + m_{p_1}^k) \cap \cdots \cap (J + m_{p_s}^k)$. Using Proposition 1 we get $fg \in J$, so that $f \in J : I$.

Now let us take $f \in J : I$. For any $g \in I$ we have $fg \in J \subset (J + I^k)$, and then $f \in (J + I^k) : I$. For $1 \leq i \leq s$ we also have

$$fg \in J + m_{p_i}^k.$$

Because $p_i \notin V(I)$, then there exists $h \in I$ such that $h(p_i) \neq 0$. From Lemma 2, h is invertible in $\mathbb{C}[x]/(J + m_{p_i}^k)$. As $fh \equiv 0$ in $\mathbb{C}[x]/(J + m_{p_i}^k)$ we have

$$f \in J + m_{p_i}^k.$$

So $f \in (\bigcap_{i=1}^s (J + m_{p_i}^k)) \cap ((J + I^k) : I)$. $\square$

From now on we shall assume that $(J : I) + I = \mathbb{C}[x]$. We have

Lemma 4. *If* $(J : I) + I = \mathbb{C}[x]$ *and* $\dim_{\mathbb{C}} \mathbb{C}[x]/(J : I) < \infty$ *then* $V(J : I) \cap V(I) = \emptyset$*, and then* $V(J : I) = V(J) \setminus V(I) = \{p_1, \ldots, p_s\}$. □

Lemma 5. *For every positive integer k we have* $(J : I) + I^k = \mathbb{C}[x]$.

Proof. We have

$$V((J : I) + I^k) = V(J : I) \cap V(I^k) = V(J : I) \cap V(I).$$

Since $(J : I) + I = \mathbb{C}[x]$, we have $V(J : I) \cap V(I) = \emptyset$. So

$$(J : I) + I^k = \mathbb{C}[x]. \quad \square$$

Lemma 6. *For every positive integer k we have* $(J + I^k) : I = \mathbb{C}[x]$.

Proof. Let us take $f \in (J : I) + I^k$. Then $f = h_1 + h_2$, where $h_1 \in J : I$ and $h_2 \in I^k$. For any $g \in I$ we get

$$fg = h_1 g + h_2 g \in J + I^k.$$

So we have $(J : I) + I^k \subseteq (J + I^k) : I$. Using the previous lemma we get

$$\mathbb{C}[x] = (J : I) + I^k \subseteq (J + I^k) : I \subseteq \mathbb{C}[x]. \quad \square$$

Lemmas 3 and 6 and Proposition 1 imply

Corollary 7. *If* $(J : I) + I = \mathbb{C}[x]$ *then for all k large enough*

$$J : I = \bigcap_{i=1}^{s} (J + m_{p_i}^k). \quad \square$$

Ideals $J + m_{p_i}^k$ are pairwise comaximal. As a consequence of the Chinese Remainder Theorem we get

Proposition 8. *If* $(J : I) + I = \mathbb{C}[x]$ *then for all k large enough*

$$\frac{\mathbb{C}[x]}{J : I} \cong \frac{\mathbb{C}[x]}{J + m_{p_1}^k} \times \cdots \times \frac{\mathbb{C}[x]}{J + m_{p_s}^k}. \quad \square$$

2. Quotients of ideals in $\mathbb{R}[x]$ and $\mathbb{C}[x]$

Lemma 9. *Let* $f_1, \ldots, f_r$ *be polynomials with real coefficients. Let* $S_{\mathbb{R}}$ *(resp.* $S_{\mathbb{C}}$*) denote the ideal in* $\mathbb{R}[x]$ *(resp.* $\mathbb{C}[x]$*) generated by* $f_1, \ldots, f_r$*. Then*

(i) $S_{\mathbb{R}} = S_{\mathbb{C}} \cap \mathbb{R}[x]$,
(ii) $S_{\mathbb{R}} = \mathbb{R}[x] \Leftrightarrow S_{\mathbb{C}} = \mathbb{C}[x]$.

Proof. (i) If $f \in S_{\mathbb{R}}$ then obviously $f \in S_{\mathbb{C}} \cap \mathbb{R}[x]$.
If $h = \sum_\alpha a_\alpha x^\alpha \in \mathbb{C}[x]$ then put $\overline{h} = \sum_\alpha \overline{a_\alpha} x^\alpha$. In that case $h \in \mathbb{R}[x]$ if and only if $h = \overline{h}$. Take $f \in S_{\mathbb{C}} \cap \mathbb{R}[x]$. There exist $h_1, \ldots, h_r \in \mathbb{C}[x]$ such that $f = \sum_{i=1}^r h_i f_i$. As $f, f_1, \ldots, f_r \in \mathbb{R}[x]$, we have $\overline{f} = f = \sum_{i=1}^r \overline{h_i} f_i$. Hence $f = \frac{1}{2} \sum_{i=1}^r (h_i + \overline{h_i}) f_i$. Of course $h_i + \overline{h_i} \in \mathbb{R}[x]$, so $f \in S_{\mathbb{R}}$.
(ii) If $S_{\mathbb{R}} = \mathbb{R}[x]$ then $1 \in S_{\mathbb{R}} \subset S_{\mathbb{C}}$, so $S_{\mathbb{C}} = \mathbb{C}[x]$.
If $S_{\mathbb{C}} = \mathbb{C}[x]$ then $S_{\mathbb{R}} = S_{\mathbb{C}} \cap \mathbb{R}[x] = \mathbb{C}[x] \cap \mathbb{R}[x] = \mathbb{R}[x]$. □

Lemma 10. *Assume that* $J_{\mathbb{R}}, I_{\mathbb{R}}$ *are ideals in* $\mathbb{R}[x]$*. Then* $(J_{\mathbb{R}} : I_{\mathbb{R}})_{\mathbb{C}} = J_{\mathbb{C}} : I_{\mathbb{C}}$*, and then* $(J_{\mathbb{C}} : I_{\mathbb{C}}) \cap \mathbb{R}[x] = J_{\mathbb{R}} : I_{\mathbb{R}}$.

Proof. Let $g_1, \ldots, g_s \in \mathbb{R}[x]$ be generators of the ideal $I_{\mathbb{R}}$, and in consequence of the ideal $I_{\mathbb{C}}$, and let $f_1, \ldots, f_l$ generate $J_{\mathbb{R}}$ and also $J_{\mathbb{C}}$. Of course

$$J_{\mathbb{R}} : I_{\mathbb{R}} = \{h \in \mathbb{R}[x] : hg_i \in J_{\mathbb{R}}, \text{ for each } 1 \le i \le s\}$$
$$J_{\mathbb{C}} : I_{\mathbb{C}} = \{h \in \mathbb{C}[x] : hg_i \in J_{\mathbb{C}}, \text{ for each } 1 \le i \le s\}.$$

Take $h \in (J_{\mathbb{R}} : I_{\mathbb{R}})_{\mathbb{C}}$, then there exist $w_1, \ldots, w_m \in \mathbb{C}[x]$ and $v_1, \ldots, v_m \in J_{\mathbb{R}} : I_{\mathbb{R}}$ such that

$$h = \sum_{j=1}^{m} w_j v_j.$$

For each $1 \le i \le s$, $v_j g_i \in J_{\mathbb{R}}$, and $hg_i \in J_{\mathbb{C}}$, so $h \in J_{\mathbb{C}} : I_{\mathbb{C}}$.

Take $h \in J_{\mathbb{C}} : I_{\mathbb{C}}$, then for each $1 \le i \le s$, $hg_i \in J_{\mathbb{C}}$, so there exist $w_1, \dots, w_l \in \mathbb{C}[x]$ such that $hg_i = \sum_{j=1}^{l} f_j w_j$. Then $\overline{h} g_i = \sum_{j=1}^{l} f_j \overline{w}_j$ and

$$(h+\overline{h})g_i = \sum_{j=1}^{l} (w_j + \overline{w}_j) f_j.$$

Of course $w_j + \overline{w}_j \in \mathbb{R}[x]$. So $(h+\overline{h})g_i \in J_{\mathbb{R}}$ for each $1 \le i \le s$, and then $h + \overline{h} \in J_{\mathbb{R}} : I_{\mathbb{R}} \subset (J_{\mathbb{R}} : I_{\mathbb{R}})_{\mathbb{C}}$. As

$$(h-\overline{h})g_i = \sum_{j=1}^{l} (w_j - \overline{w}_j) f_j$$

and $\sqrt{-1}(w_j - \overline{w}_j) \in \mathbb{R}[x]$ so $\sqrt{-1}(h-\overline{h})g_i \in J_{\mathbb{R}}$ for each $1 \le i \le s$. So we get $\sqrt{-1}(h-\overline{h}) \in J_{\mathbb{R}} : I_{\mathbb{R}}$, and then $h - \overline{h} \in (J_{\mathbb{R}} : I_{\mathbb{R}})_{\mathbb{C}}$. Because $h+\overline{h} \in (J_{\mathbb{R}} : I_{\mathbb{R}})_{\mathbb{C}}$ and $h-\overline{h} \in (J_{\mathbb{R}} : I_{\mathbb{R}})_{\mathbb{C}}$ then $h \in (J_{\mathbb{R}} : I_{\mathbb{R}})_{\mathbb{C}}$. $\square$

If $S_{\mathbb{K}}$ is an ideal in $\mathbb{K}[x]$, where $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$, denote

$$V(S_{\mathbb{K}}) = \{p \in \mathbb{K}^n : f(p) = 0 \text{ for all } f \in S_{\mathbb{K}}\}.$$

Consider ideals $J_{\mathbb{R}} \subset I_{\mathbb{R}} \subset \mathbb{R}[x]$, such that

$$\dim_{\mathbb{R}} \frac{\mathbb{R}[x]}{J_{\mathbb{R}} : I_{\mathbb{R}}} < \infty \quad \text{and} \quad (J_{\mathbb{R}} : I_{\mathbb{R}}) + I_{\mathbb{R}} = \mathbb{R}[x].$$

Then $\mathbb{C}[x] = (J_{\mathbb{C}} : I_{\mathbb{C}}) + I_{\mathbb{C}}$ and $\dim_{\mathbb{C}} \mathbb{C}[x]/(J_{\mathbb{C}} : I_{\mathbb{C}}) = \dim_{\mathbb{R}} \mathbb{R}[x]/(J_{\mathbb{R}} : I_{\mathbb{R}}) < \infty$. By Lemma 4, $V(J_{\mathbb{C}} : I_{\mathbb{C}}) = V(J_{\mathbb{C}}) \setminus V(I_{\mathbb{C}})$ is finite. Let $V(J_{\mathbb{C}}) \setminus V(I_{\mathbb{C}}) = \{p_1, \dots, p_s\}$. By Proposition 8 there exists a positive integer k such that

$$\frac{\mathbb{C}[x]}{(J_{\mathbb{C}} : I_{\mathbb{C}})} \cong \frac{\mathbb{C}[x]}{J_{\mathbb{C}} + m_{p_1}^k} \times \cdots \times \frac{\mathbb{C}[x]}{J_{\mathbb{C}} + m_{p_s}^k}.$$

Set $V(J_{\mathbb{R}}) \setminus V(I_{\mathbb{R}}) = \{p_1, \dots, p_m\}$ and $(V(J_{\mathbb{C}}) \setminus V(I_{\mathbb{C}})) \setminus \mathbb{R}^n = \{q_1, \overline{q_1}, \dots, q_r, \overline{q_r}\}$. Of course $s = m + 2r$. For k large enough and $p \in V(J_{\mathbb{R}}) \setminus V(I_{\mathbb{R}})$ we define an $\mathbb{R}$-algebra

$$\mathcal{A}_{\mathbb{R},p} := \frac{\mathbb{R}[x]}{J_{\mathbb{R}} + m_{\mathbb{R},p}^k}, \quad \text{where } m_{\mathbb{R},p} = \{f \in \mathbb{R}[x] : f(p) = 0\}.$$

For $p \in V(J_{\mathbb{C}}) \setminus V(I_{\mathbb{C}})$, we define a $\mathbb{C}$-algebra

$$\mathcal{A}_{\mathbb{C},p} := \frac{\mathbb{C}[x]}{J_{\mathbb{C}} + m_{\mathbb{C},p}^k}, \quad \text{where } m_{\mathbb{C},p} = \{f \in \mathbb{C}[x] : f(p) = 0\}.$$

Of course, $f \in J_{\mathbb{C}} + m_{\mathbb{C},p}^k$ if and only if $\overline{f} \in J_{\mathbb{C}} + m_{\mathbb{C},\overline{p}}^k$. In particular

$$\mathbb{R}[x] \cap (J_{\mathbb{C}} + m_{\mathbb{C},p}^k) = \mathbb{R}[x] \cap (J_{\mathbb{C}} + m_{\mathbb{C},\overline{p}}^k). \tag{2}$$

The mapping $f \longmapsto \overline{f}$ induces an $\mathbb{R}$–isomorphism of algebras $\mathcal{A}_{\mathbb{C},p}$ and $\mathcal{A}_{\mathbb{C},\overline{p}}$, so that

$$\dim_{\mathbb{C}} \mathcal{A}_{\mathbb{C},p} = 2\dim_{\mathbb{R}} \mathcal{A}_{\mathbb{C},p} = 2\dim_{\mathbb{R}} \mathcal{A}_{\mathbb{C},\overline{p}} = \dim_{\mathbb{C}} \mathcal{A}_{\mathbb{C},\overline{p}}. \tag{3}$$

Let us denote

$$\mathcal{B} = \bigoplus_{i=1}^{m} \mathcal{A}_{\mathbb{R},p_i} \bigoplus_{j=1}^{r} \mathcal{A}_{\mathbb{C},q_j}.$$

Theorem 11. *For all k large enough there is a natural isomorphism*

$$\frac{\mathbb{R}[x]}{J_{\mathbb{R}} : I_{\mathbb{R}}} \cong \mathcal{B}.$$

Proof. We take k as in Proposition 8. We define a homomorphism

$$\pi : \mathbb{R}[x] \longrightarrow \mathcal{B},$$

as $\pi(f) = \oplus_{i=1}^{m} [f]_{p_i} \oplus_{j=1}^{r} [f]_{q_j}$, where $[f]_p$ is the residue class of f in the appropriate algebra. Then

$$\pi^* : \frac{\mathbb{R}[x]}{\ker \pi} \longrightarrow \mathcal{B}$$

is a monomorphism. Using Corollary 7 and (2) we get

$$\begin{aligned}\ker\pi &= \mathbb{R}[x]\cap\bigcap_{i=1}^{m}(J_{\mathbb{R}}+m_{\mathbb{R},p_i}^k)\cap\bigcap_{j=1}^{r}(J_{\mathbb{C}}+m_{\mathbb{C},q_j}^k)\\ &= \mathbb{R}[x]\cap\bigcap_{i=1}^{m}(J_{\mathbb{R}}+m_{\mathbb{R},p_i}^k)_{\mathbb{C}}\cap\bigcap_{j=1}^{r}(J_{\mathbb{C}}+m_{\mathbb{C},q_j}^k)\\ &= \mathbb{R}[x]\cap\bigcap_{i=1}^{m}(J_{\mathbb{C}}+m_{\mathbb{C},p_i}^k)\cap\bigcap_{j=1}^{r}(J_{\mathbb{C}}+m_{\mathbb{C},q_j}^k)\cap\bigcap_{j=1}^{r}(J_{\mathbb{C}}+m_{\mathbb{C},\bar{q}_j}^k)\\ &= \mathbb{R}[x]\cap(J_{\mathbb{C}}:I_{\mathbb{C}})=J_{\mathbb{R}}:I_{\mathbb{R}}.\end{aligned}$$

By (3) we have

$$\begin{aligned}\dim_{\mathbb{R}}\mathcal{B} &= \sum_{i=1}^{m}\dim_{\mathbb{R}}\mathcal{A}_{\mathbb{R},p_i}+\sum_{j=1}^{r}\dim_{\mathbb{R}}\mathcal{A}_{\mathbb{C},q_j}\\ &= \sum_{i=1}^{m}\dim_{\mathbb{C}}\mathcal{A}_{\mathbb{C},p_i}+2\cdot\sum_{j=1}^{r}\dim_{\mathbb{C}}\mathcal{A}_{\mathbb{C},q_j}\\ &= \sum_{i=1}^{m}\dim_{\mathbb{C}}\mathcal{A}_{\mathbb{C},p_i}+\sum_{j=1}^{r}\dim_{\mathbb{C}}\mathcal{A}_{\mathbb{C},q_j}+\sum_{j=1}^{r}\dim_{\mathbb{C}}\mathcal{A}_{\mathbb{C},\bar{q}_j}\,.\end{aligned}$$

By Proposition 8, it equals

$$\dim_{\mathbb{C}}\frac{\mathbb{C}[x]}{J_{\mathbb{C}}:I_{\mathbb{C}}}=\dim_{\mathbb{R}}\frac{\mathbb{R}[x]}{J_{\mathbb{R}}:I_{\mathbb{R}}}.$$

So $\dim_{\mathbb{R}}\mathcal{B}=\dim_{\mathbb{R}}\mathbb{R}[x]/(J_{\mathbb{R}}:I_{\mathbb{R}})$ and then $\pi^*:\mathbb{R}[x]/(J_{\mathbb{R}}:I_{\mathbb{R}})\to\mathcal{B}$ is an isomorphism. □

3. Bilinear forms

Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. For $p\in\mathbb{K}^n$, let $\mathcal{O}_{\mathbb{K},p}$ denote the ring of germs at p of analytic functions $\mathbb{K}^n\longrightarrow\mathbb{K}$. There is a natural homomorphism

$$\eta:\mathbb{K}[x]=\mathbb{K}[x_1,\ldots,x_n]\longrightarrow\mathcal{O}_{\mathbb{K},p}.$$

Let $m_{\mathbb{K},p}=\{f\in\mathbb{K}[x]:f(p)=0\}$ be the maximal ideal in $\mathbb{K}[x]$ associated with p.

Let $S_{\mathbb{R}}$ be an ideal in $\mathbb{R}[x]$, let $S_{\mathbb{C}}$ denote the ideal in $\mathbb{C}[x]$ generated by $S_{\mathbb{R}}$, and let

$$V(S_{\mathbb{K}})=\{p\in\mathbb{K}^n:f(p)=0\text{ for all }f\in S_{\mathbb{K}}\}.$$

Take $h_1,\ldots,h_n\in S_{\mathbb{R}}$, the ideal $J_{\mathbb{K}}$ generated by $h_1,\ldots,h_n$, and a polynomial mapping $H_{\mathbb{K}}=(h_1,\ldots,h_n):\mathbb{K}^n\longrightarrow\mathbb{K}^n$. Then $J_{\mathbb{K}}\subset S_{\mathbb{K}}$ and $V(S_{\mathbb{K}})\subset V(J_{\mathbb{K}})=H_{\mathbb{K}}^{-1}(0)$. In particular, points isolated in $H_{\mathbb{C}}^{-1}(0)$ are isolated in $V(S_{\mathbb{K}})$.

Take $p\in V(S_{\mathbb{K}})$. Let $S_{\mathbb{K},p}$ (resp. $J_{\mathbb{K},p}$) denote the ideal in $\mathcal{O}_{\mathbb{K},p}$ generated by $S_{\mathbb{K}}$ (resp. $J_{\mathbb{K}}$), and let $\mathcal{A}'_{\mathbb{K},p}=\mathcal{O}_{\mathbb{K},p}/S_{\mathbb{K},p}$. Clearly $\mathcal{A}'_{\mathbb{K},p}$ is a $\mathbb{K}$-algebra and $\eta(S_{\mathbb{K}})\subset S_{\mathbb{K},p}$.

Assume that $S_{\mathbb{K},p}=J_{\mathbb{K},p}$ so that $\mathcal{A}'_{\mathbb{K},p}=\mathcal{O}_{\mathbb{K},p}/J_{\mathbb{K},p}$. Then $\dim_{\mathbb{K}}\mathcal{A}'_{\mathbb{K},p}<\infty$ if and only if p is isolated in $H_{\mathbb{C}}^{-1}(0)$. If that is the case then p is isolated in $V(S_{\mathbb{R}})$, and $\eta(m_{\mathbb{K},p}^k)\subset S_{\mathbb{K},p}$ for all k large enough.

Lemma 12. *If $p\in V(S_{\mathbb{K}})$ is isolated in $H_{\mathbb{C}}^{-1}(0)$ and $S_{\mathbb{K},p}=J_{\mathbb{K},p}$, then η induces an isomorphism of $\mathbb{K}$-algebras*

$$\eta:\mathbb{K}[x]/(S_{\mathbb{K}}+m_{\mathbb{K},p}^k)\longrightarrow\mathcal{A}'_{\mathbb{K},p}$$

for all k large enough. □

For p and k as above, put

$$\mathcal{A}_{\mathbb{K},p}=\mathbb{K}[x]/(S_{\mathbb{K}}+m_{\mathbb{K},p}^k),$$

so that $\mathcal{A}_{\mathbb{K},p}$ is isomorphic to $\mathcal{A}'_{\mathbb{K},p}$. In particular, $\mathcal{A}_{\mathbb{K},p}$ does not depend on k, if k is large enough, and $\dim_{\mathbb{K}}\mathcal{A}_{\mathbb{K},p}<\infty$.

Applying the formula for the local topological degree by Eisenbud and Levine [5] and Khimshiashvili [6,7], and the theory of Frobenius algebras (see [8–10]) one may prove some properties, presented in [1], of bilinear forms on the algebra $\mathcal{O}_{\mathbb{K},p}/J_{\mathbb{K},p}=\mathcal{A}'_{\mathbb{K},p}$. As $\mathcal{A}'_{\mathbb{K},p}$ is isomorphic to $\mathcal{A}_{\mathbb{K},p}$, they hold true for the algebra $\mathcal{A}_{\mathbb{K},p}$.

Let points $p_1, \ldots, p_w \in V(S_{\mathbb{C}})$ be as in Lemma 12. One may assume that $p_1, \ldots, p_m \in \mathbb{R}^n$ and $p_{m+1}, \ldots, p_w \in \mathbb{C}^n \setminus \mathbb{R}^n$. Denote

$$\mathcal{B} = \bigoplus_{i=1}^{m} \mathcal{A}_{\mathbb{R},p_i} \oplus_{j=m+1}^{w} \mathcal{A}_{\mathbb{C},p_j}.$$

Obviously, $\mathcal{B}$ is a finite dimensional $\mathbb{R}$-algebra.

Let $u \in \mathbb{R}[x]$, and let $\varphi : \mathcal{B} \longrightarrow \mathbb{R}$ be a linear functional. Then there are bilinear symmetric forms $\Phi,\ \Psi : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{R}$ given by $\Phi(a, b) = \varphi(ab)$ and $\Psi(a, b) = \varphi(uab)$.

One may define signature Φ, and similarly signature Ψ, as the dimension of a maximal subspace of $\mathcal{B}$ on which Φ is positive definite minus the dimension of that one on which Φ is negative definite.

Let $\det[\Phi]$ (resp. $\det[\Psi]$) denote the determinant of the matrix of Φ (resp. Ψ), with respect to some basis of $\mathcal{B}$. The sign of the determinant does not depend on the choice of a basis.

Theorem 13 (*[1, Theorem 2.3, p. 306]*). *Let points $p_1, \ldots, p_w \in V(S_{\mathbb{C}}) \subset H_{\mathbb{C}}^{-1}(0)$ be as in Lemma 12. Suppose that $\det[\Psi] \neq 0$. Then $\det[\Phi] \neq 0$, $u(p_i) \neq 0$ for $1 \le i \le m$, and*

$$\sum \deg_p H_{\mathbb{R}} \equiv \dim_{\mathbb{R}} \mathcal{B} + 1 + (\operatorname{sgn}\det[\Phi] + \operatorname{sgn}\det[\Psi])/2 \pmod 2,$$

where $p \in \{p_1, \ldots, p_m\} \cap \{u > 0\}$ and $\deg_p H_{\mathbb{R}}$ denotes the local topological degree of $H_{\mathbb{R}}$ at p. □

For $x = (x_1, \ldots, x_n),\ x' = (x'_1, \ldots, x'_n)$, and $1 \le i, j \le n$ define

$$T_{ij}(x, x') = \frac{h_i(x'_1, \ldots, x'_{j-1}, x_j, \ldots, x_n) - h_i(x'_1, \ldots, x'_j, x_{j+1}, \ldots, x_n)}{x_j - x'_j}.$$

It is easy to see that each T_{ij} extends to a polynomial, thus we may assume that

$$T_{ij} \in \mathbb{R}[x, x'] = \mathbb{R}[x_1, \ldots, x_n, x'_1, \ldots, x'_n].$$

There is the natural projection $\mathbb{R}[x, x'] \longrightarrow \mathcal{B} \otimes \mathcal{B}$ given by

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} (x'_1)^{\beta_1} \cdots (x'_n)^{\beta_n} \mapsto x_1^{\alpha_1} \cdots x_n^{\alpha_n} \otimes (x'_1)^{\beta_1} \cdots (x'_n)^{\beta_n}.$$

Let T denote the image of $\det[T_{ij}(x, x')]$ in $\mathcal{B} \otimes \mathcal{B}$.

Put $d = \dim_{\mathbb{R}} \mathcal{B}$. Assume that $e_1, \ldots, e_d$ form a basis in $\mathcal{B}$. So $\dim_{\mathbb{R}} \mathcal{B} \otimes \mathcal{B} = d^2$ and $e_i \otimes e_j$, for $1 \le i, j \le d$, form a basis in $\mathcal{B} \otimes \mathcal{B}$. Hence there are $t_{ij} \in \mathbb{R}$ such that

$$T = \sum_{i,j=1}^{d} t_{ij} e_i \otimes e_j = \sum_{i=1}^{d} e_i \otimes \hat{e}_i,$$

where $\hat{e}_i = \sum_{j=1}^{d} t_{ij} e_j$. Elements $\hat{e}_1, \ldots, \hat{e}_d$ form a basis in $\mathcal{B}$. So there are $A_1, \ldots, A_d \in \mathbb{R}$ such that

$$1 = A_1 \hat{e}_1 + \cdots + A_d \hat{e}_d \text{ in } \mathcal{B}.$$

Definition. For $f = a_1 e_1 + \cdots + a_d e_d \in \mathcal{B}$ define $\varphi_T(f) = a_1 A_1 + \cdots + a_d A_d$. Hence $\varphi_T : \mathcal{B} \longrightarrow \mathbb{R}$ is a linear functional.

Let Φ_T be the bilinear form on $\mathcal{B}$ given by $\Phi_T(a, b) = \varphi_T(ab)$.

Theorem 14 (*[1, Theorem 1.5, p. 304]*). *The form Φ_T is non-degenerate and*

$$\sum_{i=1}^{m} \deg_{p_i} H_{\mathbb{R}} = \text{signature } \Phi_T. \quad \square$$

4. Topological degree

Let $h_1, \ldots, h_n \in \mathbb{R}[x_1, \ldots, x_n]$, and let $H_{\mathbb{K}} = (h_1, \ldots, h_n) : \mathbb{K}^n \longrightarrow \mathbb{K}^n$. Denote by $J_{\mathbb{K}}$ the ideal in $\mathbb{K}[x]$ generated by $h_1, \ldots, h_n$, so that $H_{\mathbb{C}}^{-1}(0) = V(J_{\mathbb{C}})$.

Assume that there is an ideal $I_{\mathbb{R}}$ such that $J_{\mathbb{C}} \subset I_{\mathbb{C}}$, $\dim_{\mathbb{R}} \mathbb{R}[x]/(J_{\mathbb{R}} : I_{\mathbb{R}}) < \infty$, and $(J_{\mathbb{R}} : I_{\mathbb{R}}) + I_{\mathbb{R}} = \mathbb{R}[x]$.

Put $S_{\mathbb{R}} = J_{\mathbb{R}} : I_{\mathbb{R}}$. Hence $V(S_{\mathbb{C}})$ is finite, and by Lemma 4

$$H_{\mathbb{C}}^{-1}(0) \setminus V(I_{\mathbb{C}}) = V(J_{\mathbb{C}}) \setminus V(I_{\mathbb{C}}) = V(S_{\mathbb{C}}).$$

Hence each $p \in V(J_{\mathbb{C}}) \setminus V(I_{\mathbb{C}})$ is isolated in $H_{\mathbb{C}}^{-1}(0)$. By Corollary 7, if k is large enough then the ideal $S_{\mathbb{K},p} \subset \mathcal{O}_{\mathbb{K},p}$ generated by $S_{\mathbb{R}}$ equals the ideal generated by $J_{\mathbb{K}} + m_{\mathbb{K},p}^k$.

Let $J_{\mathbb{K},p}$ denote the ideal in $\mathcal{O}_{\mathbb{K},p}$ generated by $J_{\mathbb{R}}$. Since $J_{\mathbb{K},p}$ has an algebraically isolated zero at p, the local Nullstellensatz implies that $m^k_{\mathbb{K},p} \subset J_{\mathbb{K},p}$, so that $S_{\mathbb{K},p} = J_{\mathbb{K},p} + m^k_{\mathbb{K},p} = J_{\mathbb{K},p}$. Hence each point $p \in V(S_{\mathbb{C}}) = H_{\mathbb{C}}^{-1}(0) \setminus V(I_{\mathbb{C}})$ satisfies the assumptions of Lemma 12.

Put $H_{\mathbb{R}}^{-1}(0) \setminus V(I_{\mathbb{R}}) = \{p_1, \ldots, p_m\}$ and

$$(H_{\mathbb{C}}^{-1}(0) \setminus V(I_{\mathbb{C}})) \setminus \mathbb{R}^n = \{q_1, \overline{q_1}, \ldots, q_r, \overline{q_r}\} .$$

By Theorem 11, $\mathcal{A} := \mathbb{R}[x]/S = \mathbb{R}[x]/(J_{\mathbb{R}} : I_{\mathbb{R}})$ and

$$\mathcal{B} = \bigoplus_{i=1}^{m} \mathcal{A}_{\mathbb{R},p_i} \oplus_{j=1}^{r} \mathcal{A}_{\mathbb{C},q_j}$$

are isomorphic.

As a consequence of Theorem 13 we get

Theorem 15. *Assume that* $\dim_{\mathbb{R}} \mathcal{A} < \infty$, $u \in \mathbb{R}[x]$ *and* $\varphi : \mathcal{A} \to \mathbb{R}$ *is a linear functional. Let bilinear symmetric forms* $\Phi, \Psi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{R}$ *be given by* $\Phi(a, b) = \varphi(ab)$ *and* $\Psi(a, b) = \varphi(uab)$.

Suppose that $\det[\Psi] \neq 0$. *Then* $\det[\Phi] \neq 0$, $u(p) \neq 0$ *for each* $p \in H_{\mathbb{R}}^{-1}(0) \setminus V(I_{\mathbb{R}})$, *and*

$$\sum \deg_p H_{\mathbb{R}} \equiv \dim_R \mathcal{A} + 1 + (\operatorname{sgn}\det[\Phi] + \operatorname{sgn}\det[\Psi])/2 \pmod 2,$$

where $p \in H_{\mathbb{R}}^{-1}(0) \cap \{u > 0\} \setminus V(I_{\mathbb{R}})$. □

In the same way as in Section 3 one may construct the bilinear symmetric form Φ_T on $\mathcal{A} = \mathbb{R}[x]/(J_{\mathbb{R}} : I_{\mathbb{R}})$. As a consequence of Theorem 14 we get

Theorem 16. *The form* Φ_T *is non-degenerate and*

$$\text{signature } \Phi_T = \sum \deg_p H_{\mathbb{R}} ,$$

where $p \in H_{\mathbb{R}}^{-1}(0) \setminus V(I_{\mathbb{R}})$. □

Let Ψ_T be the bilinear form on $\mathcal{A}$ given by $\Psi_T(a, b) = \varphi_T(uab)$. Using the same arguments as in [1, Theorem 1.5, p. 304] one may prove

Theorem 17. Ψ_T *is non-degenerate if and only if* $u(p) \neq 0$ *for each* $p \in H_{\mathbb{C}}^{-1}(0) \setminus V(I_{\mathbb{C}})$. *If that is the case then*

$$\sum \deg_p H_{\mathbb{R}} = \frac{1}{2}(\text{signature } \Phi_T + \text{signature } \Psi_T),$$

where $p \in H_{\mathbb{R}}^{-1}(0) \cap \{u > 0\} \setminus V(I_{\mathbb{R}})$.

5. Immersions

Let M be an m-dimensional manifold. A C^1 map $g : M \longrightarrow \mathbb{R}^k$ is called an immersion, if for each $p \in M$ the rank of $Dg(p)$ equals m.

Assume that m is even, $k = 2m$ and M is compact and oriented. Define

$$G : M \times M \longrightarrow \mathbb{R}^{2m} \quad \text{as} \quad G(x, y) = g(x) - g(y).$$

Set $\Delta = \{(p, p) : p \in M\} \subset M \times M$. Since g is an immersion, Δ is isolated in $G^{-1}(0)$, and so $G^{-1}(0) \setminus \Delta$ is a compact subset of $M \times M \setminus \Delta$. There exists $(N, \partial N)$ — a compact $2m$-dimensional oriented manifold with a boundary, such that

$$N \subset M \times M \setminus \Delta \quad \text{and} \quad G^{-1}(0) \setminus \Delta \subset N \setminus \partial N.$$

Denote by $d(G)$ the topological degree of the mapping

$$\partial N \ni (x, y) \mapsto \frac{G(x, y)}{\|G(x, y)\|} \in S^{2m-1}.$$

Of course, $d(G)$ does not depend on the choice of N. In particular, if $G^{-1}(0) \setminus \Delta$ is finite then

$$d(G) = \sum \deg_z G, \quad \text{where } z \in G^{-1}(0) \setminus \Delta,$$

and $\deg_z G$ denotes the local topological degree of G at z.

Whitney has introduced in [2] an intersection number $I(g)$ for the immersion g. According to the Lashof and Smale [11, Theorem 3.1],

$$d(G) = 2I(g).$$

Now assume that $f = (f_1, \ldots f_n) : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ is a C^1 mapping, such that $M = f^{-1}(0)$ and M is a complete intersection, i.e. for each $p \in M$ the rank of $Df(p)$ equals n.

We shall say that vectors $v_1, \dots, v_m \in T_pM$ are well oriented if vectors $\nabla f_1(p), \dots, \nabla f_n(p), v_1, \dots, v_m$ are well oriented in $\mathbb{R}^{n+m}$. Put

$$F(x, y) = (f(x), f(y)) : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n \times \mathbb{R}^n.$$

Then $F^{-1}(0) = M \times M$ is a complete intersection. The orientation of $F^{-1}(0)$ defined the way as above is the same as the orientation of the product $M \times M$.

Let

$$\overline{g} = (g_1, \dots, g_{2m}) : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^{2m}$$

be a C^1 mapping. Put $g = \overline{g}|_M$. It easy to verify that

$$\operatorname{rank}\left[Dg(p)\right] = \operatorname{rank}\begin{bmatrix} D\overline{g}(p) \\ Df(p) \end{bmatrix} - n$$

at each point $p \in M$, so we have

Lemma 18. *$g = \overline{g}|_M : M \longrightarrow \mathbb{R}^{2m}$ is an immersion if and only if at each point $p \in M$*

$$\operatorname{rank}\begin{bmatrix} D\overline{g}(p) \\ Df(p) \end{bmatrix} = n + m. \quad \square$$

Example 19. Let $f = x^2 + y^2 + z^2 - r^2$, and

$$\overline{g} = (x, y, xz, yz) : \mathbb{R}^3 \longrightarrow \mathbb{R}^4,$$

i.e. $m = 2$, $n = 1$. Then $M = f^{-1}(0)$ is the 2-dimensional sphere $S^2(r)$ of radius r. As

$$\operatorname{rank}\begin{bmatrix} D\overline{g} \\ Df \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z & 0 & x \\ 0 & z & y \\ 2x & 2y & 2z \end{bmatrix}$$

has a non-zero (3×3)–minor at each point $p \in \mathbb{R}^3 \setminus \{0\}$, then $g = \overline{g}|_{S^2(r)}$ is an immersion for every $r > 0$.

Let

$$\overline{G}(x, y) = \overline{g}(x) - \overline{g}(y).$$

Then $G(x, y) = g(x) - g(y) = \overline{G}|_{M\times M} = \overline{G}|_{F^{-1}(0)}$. Put

$$H(x, y) = (F(x, y), \overline{G}(x, y)) = (f_1(x), \dots, f_n(x), f_1(y), \dots, f_n(y), g_1(x) - g_1(y), \dots, g_{2m}(x) - g_{2m}(y)).$$

Then $H : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^{2n+2m}$, and $(p, q) \in H^{-1}(0)$ if and only if $(p, q) \in M \times M$ and $G(p, q) = 0$. By [1, Lemma 3.2], $z = (p, q) \in M \times M$ is isolated in $G^{-1}(0)$ if and only if z is isolated in $H^{-1}(0)$, and if that is the case then

$$\deg_z G = \deg_z H.$$

Let $\Delta = \{(p, p) | p \in \mathbb{R}^{n+m}\}$ be the diagonal in $\mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$. Then

$$z = (p, q) \in H^{-1}(0) \setminus \Delta \quad \text{if and only if} \quad z = (p, q) \in G^{-1}(0) \setminus \Delta.$$

So we get

Proposition 20. *Suppose that m is even, and*

(a) *$M = f^{-1}(0)$ is an oriented compact m-dimensional complete intersection,*
(b) *$g = \overline{g}|_M : M \longrightarrow \mathbb{R}^{2m}$ is an immersion,*
(c) *$H^{-1}(0) \setminus \Delta$ is finite.*

Then $2I(g) = d(G) = \Sigma \deg_z H$, where $z \in H^{-1}(0) \setminus \Delta$. $\square$

A homotopy $h_t : M \longrightarrow \mathbb{R}^{2m}$ is called *a regular homotopy*, if at each stage it is an immersion and the induced homotopy of the tangent bundle is continuous.

As in [2] we say that an immersion $g : M \longrightarrow \mathbb{R}^{2m}$ has *a regular self-intersection* at the point $g(p) = g(q)$ if

$$Dg(p)T_pM + Dg(q)T_qM = \mathbb{R}^{2m}.$$

That is so if and only if $\det[DH(p, q)] \neq 0$.

If g has only regular self-intersections, and there are no triple points $g(p) = g(q) = g(w)$, then we say that g is completely regular. If m is odd, then one can define *the intersection number* of a completely regular immersion as the number of its self-intersections modulo 2.

By [2, Theorem 2], if M is compact then the intersection number is invariant under regular homotopies. As in [12, Section 8], for any immersion $g\colon M \longrightarrow \mathbb{R}^{2m}$ there exists a completely regular immersion $\tilde{g}\colon M \longrightarrow \mathbb{R}^{2m}$ which is arbitrarily close to g in C^1-topology and there is a regular homotopy between g and $\tilde{g}$. Thus if $m > 1$ is odd then one can define the intersection number $I(g)$ as the number of self-intersections of $\tilde{g}$ modulo 2.

Suppose that g has a finite number of self-intersections, i.e. $H^{-1}(0) \setminus \Delta$ is finite. The immersion has a self-intersection $g(p) = g(q)$ if and only if (p, q) and (q, p) belong to $H^{-1}(0) \setminus \Delta$. There exists a linear function

$$u(x, y) = a_1(x_1 - y_1) + \cdots + a_{n+m}(x_{n+m} - y_{n+m})$$

which does not vanish at any point in $H^{-1}(0) \setminus \Delta$, so that $u(p, q) > 0$ if and only if $u(q, p) < 0$. Then each self-intersection is represented by a single point in $H^{-1}(0) \cap \{u > 0\} \setminus \Delta = H^{-1}(0) \cap \{u > 0\}$.

Proposition 21. *Suppose that $m > 1$ is odd, and*

(a) $M = f^{-1}(0)$ *is an oriented, compact m-dimensional complete intersection,*
(b) $g = \overline{g}|M\colon M \longrightarrow \mathbb{R}^{2m}$ *is an immersion,*
(c) $H^{-1}(0) \setminus \Delta$ *is finite,*
(d) $u(x, y) = a_1(x_1 - y_1) + \cdots + a_{n+m}(x_{n+m} - y_{n+m})$ *is such that $u(p, q) \neq 0$ for each $(p, q) \in H^{-1}(0) \setminus \Delta$.*

Then

$$I(g) \equiv \sum \deg_z H \bmod 2,$$

where $z \in H^{-1}(0) \cap \{u > 0\}$.

Proof. Put

$$\tilde{H}(x, y) = (f_1(x), \ldots, f_n(x), f_1(y), \ldots, f_n(y), \tilde{g}_1(x) - \tilde{g}_1(y), \ldots, \tilde{g}_{2m}(x) - \tilde{g}_{2m}(y)).$$

Let $B_z \subset \mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$ denote a small ball centered at $z \in H^{-1}(0) \cap \{u > 0\}$. If $\tilde{g}$ is close enough to g, then self-intersections of $\tilde{g}$ are represented by points in the set $\tilde{H}^{-1}(0) \cap \{u > 0\}$, which is a subset of the union of all B_z. As self-intersections of $\tilde{g}$ are regular,

$$I(\tilde{g}) \equiv \sum_w \operatorname{sgn} \det[D\tilde{H}(w)] \bmod 2,$$

where $w \in \tilde{H}^{-1}(0) \cap \{u > 0\}$. Each point w belongs to some B_z, hence

$$I(g) = I(\tilde{g}) \equiv \sum_z \sum_w \operatorname{sgn} \det[D\tilde{H}(w)] \bmod 2,$$

where $z \in H^{-1}(0) \cap \{u > 0\}$ and $w \in \tilde{H}^{-1}(0) \cap B_z$. Since $\tilde{H}$ is close to H in a neighborhood of $M \times M$,

$$I(g) \equiv \sum_z \deg_z H \bmod 2. \quad \square$$

If $f_1, \ldots, f_n, g_1, \ldots, g_{2m}$ are polynomials then $H = (h_1, \ldots, h_{2n+2m})$ is a polynomial mapping. Let $J_{\mathbb{R}}$ denote the ideal in $\mathbb{R}[x, y] = \mathbb{R}[x_1, \ldots, x_{n+m}, y_1, \ldots, y_{n+m}]$ generated by $h_1, \ldots, h_{2n+2m}$, and $I_{\mathbb{R}}$ the one generated by

$$f_1(x), \ldots, f_n(x), f_1(y), \ldots, f_n(y), x_1 - y_1, \ldots, x_{n+m} - y_{n+m}.$$

It is easy to verify that $J_{\mathbb{R}} \subset I_{\mathbb{R}}$. Then

$$V(J_{\mathbb{R}}) = H^{-1}(0), \; V(I_{\mathbb{R}}) = M \times M \cap \Delta.$$

Let $Q = \mathbb{R}[x, y]/J_{\mathbb{R}}$. If $m \geq 1$ and $M \neq \emptyset$, then $M \times M \cap \Delta \subset H^{-1}(0)$ is infinite, so $\dim_{\mathbb{R}} Q = \infty$ and we cannot apply methods presented in [1] so as to compute $\sum_p \deg_p H$.

Let $\mathcal{A} = \mathbb{R}[x, y]/(J_{\mathbb{R}} : I_{\mathbb{R}})$. Suppose that $\dim_{\mathbb{R}} \mathcal{A} < \infty$ and $(J_{\mathbb{R}} : I_{\mathbb{R}}) + I_{\mathbb{R}} = \mathbb{R}[x, y]$. Then $H^{-1}(0) \setminus \Delta$ is finite, so that the immersion g has a finite set of self-intersections.

Let $\Phi_T : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{R}$ be the bilinear form constructed as in Section 3. As a consequence of Theorem 16 and Proposition 20 we get

Theorem 22. *If m is even, then*

$$I(g) = \frac{1}{2} \text{ signature } \Phi_T. \quad \square$$

Take any polynomial $u(x, y) = a_1(x_1 - y_1) + \cdots + a_{n+m}(x_{n+m} - y_{n-m})$ and any linear from $\varphi : \mathcal{A} \longrightarrow \mathbb{R}$. Let $\Phi, \Psi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{R}$ be bilinear forms given by $\Phi(a, b) = \varphi(ab)$, $\Psi(a, b) = \varphi(uab)$. As a consequence of Theorem 15 and Proposition 21 we get

Theorem 23. *If* $m > 1$ *is odd and* $\det[\Psi] \neq 0$, *then*

$$I(g) \equiv \dim_{\mathbb{R}} \mathcal{A} + 1 + (\operatorname{sgn}\det[\Phi] + \operatorname{sgn}\det[\Psi])/2 \bmod 2. \quad \square$$

Example 24. Let us consider the mapping

$$g = (g_1, g_2, g_3, g_4) = (x_1, x_2, x_1x_3, x_2x_3) : \mathbb{R}^3 \longrightarrow \mathbb{R}^4.$$

As in Example 19, $g|_{S^2(1)}$ is an immersion. Put $f = x_1^2 + x_2^2 + x_3^2 - 1$. With the immersion $g|_{S^2(1)}$ we may associate the polynomial mapping

$$H = (h_1, \dots, h_6) : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^6$$

given by $h_1 = f(x_1, x_2, x_3)$, $h_2 = f(y_1, y_2, y_3)$ and $h_i = g_{i-2}(x_1, x_2, x_3) - g_{i-2}(y_1, y_2, y_3)$, for $i = 3, 4, 5, 6$.

Let $I_{\mathbb{R}}$ be the ideal in $\mathbb{R}[x_1, x_2, x_3, y_1, y_2, y_3]$ generated by $f(x), f(y), x_1 - y_1, x_2 - y_2, x_3 - y_3$ and let $J_{\mathbb{R}}$ be generated by $h_1, \dots, h_6$. Using Singular – a computer algebra system for polynomial computations – one may check that $J_{\mathbb{R}} : I_{\mathbb{R}}$ is generated by $x_1, x_2, y_1, y_2, x_3 + y_3, y_3^2 - 1$, and then monomials $e_1 = 1$ and $e_2 = y_3$ form a basis of $\mathcal{A} = \mathbb{R}[x_1, \dots, y_3]/(J_{\mathbb{R}} : I_{\mathbb{R}})$. Then $\dim_{\mathbb{R}} \mathcal{A} = 2 < \infty$ and $(J_{\mathbb{R}} : I_{\mathbb{R}}) + I_{\mathbb{R}} = \mathbb{R}[x_1, x_2, x_3, y_1, y_2, y_3]$. One may check that

$$T = -8y_3y_3' - 8 \quad \text{in } \mathcal{A} \otimes \mathcal{A}.$$

So $\hat{e}_1 = -8$ and $\hat{e}_2 = -8y'$. Clearly $1 = (-\frac{1}{8})\hat{e}_1 + 0\hat{e}_2$ in $\mathcal{A}$. Then $A_1 = -\frac{1}{8}, A_2 = 0$, so for any $a = a_1 + a_2y_3 = a_1e_1 + a_2e_2$ in $\mathcal{A}$, φ_T is given by $\varphi_T(a) = -\frac{a_1}{8}$. Then the matrix of Φ_T is given by

$$\begin{bmatrix} -\frac{1}{8} & 0 \\ 0 & -\frac{1}{8} \end{bmatrix},$$

so signature $\Phi_T = -2$, and as a consequence of Theorem 22 we get $I(g|_{S^2(1)}) = -1$.

Example 25. Let $g = (g_1, \dots, g_6) = (x_1, x_2, x_1x_3, x_2x_3, x_4, x_3x_4) : \mathbb{R}^4 \longrightarrow \mathbb{R}^6$. Using Lemma 18 it is easy to check that $g|_{S^3(1)}$ is an immersion. Put $f = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 1$ and

$$H(x_1, \dots, y_4) = (h_1, \dots, h_8) = (f(x_1, \dots, x_4), f(y_1, \dots, y_4), g(x_1, \dots, x_4) - g(y_1, \dots, y_4)).$$

Let $I_{\mathbb{R}} \subset \mathbb{R}[x_1, \dots, y_4]$ be the ideal generated by $f(x), f(y), x_1 - y_1, x_2 - y_2, x_3 - y_3, x_4 - y_4$, and $J_{\mathbb{R}}$ generated by $h_1, \dots, h_8$. One may check, using Singular, that $x_1, x_2, x_4, y_1, y_2, y_4, x_3 + y_3, y_3^2 - 1$ generate $J_{\mathbb{R}} : I_{\mathbb{R}}$, and that monomials $e_1 = 1,\ e_2 = y_3$ form the basis of $\mathcal{A} = \mathbb{R}[x_1, \dots, y_4]/(J_{\mathbb{R}} : I_{\mathbb{R}})$. Then $\dim_{\mathbb{R}} \mathcal{A} = 2 < \infty$ and $(J_{\mathbb{R}} : I_{\mathbb{R}}) + I_{\mathbb{R}} = \mathbb{R}[x_1, \dots, y_4]$. Put $\varphi : \mathcal{A} \longrightarrow \mathbb{R}$ as

$$\varphi(a) = \varphi(a_1e_1 + a_2e_2) = a_2.$$

Take $u = x_3 - y_3$. Matrices of Φ and Ψ are given by

$$\begin{bmatrix} 0 & +1 \\ +1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Of course $\det[\Psi] \neq 0$, and as consequence of Theorem 23 we get $I(g|_{S^3(1)}) \equiv 2 + 1 + \frac{1}{2}(-1 + 1) \equiv 1 \bmod 2$.

Using similar methods we have computed some more difficult examples:

Example 26.

$$h(x_1, x_2, x_3) = (2x_1x_2 + x_2, 2x_1x_3 + 4x_3, 4x_3^2 + 5x_2, 5x_2^2 + 4x_3)$$

is an immersion on the 2-dimensional sphere of radius $r = 10$. In that case $\dim_{\mathbb{R}} \mathcal{A} = 16$, and $I(h|_{S^2(10)}) = 0$.

Example 27.

$$h(x_1, x_2, x_3) = (5x_2x_3 + x_3^2 + 3x_1, 4x_1^2 + 3x_3^2 + x_3, 2x_2^2 + 3x_2x_3 + 2x_1, x_2x_3 + 4x_3^2 + 3x_2)$$

is an immersion on the 2-dimensional spheres of radius $r = 1, 10$. In both cases $\dim_{\mathbb{R}} \mathcal{A} = 6$, and $I(h|_{S^2(1)}) = 0$, $I(h|_{S^2(10)}) = 1$.

Example 28.

$$h(x_1, x_2, x_3) = (3x_1x_2 + 2x_2^2 + 2x_1x_3 + 3x_1 + 5x_3, 2x_1x_2 + 5x_2^2 + 3x_2x_3 + x_1 + 2x_2,$$
$$4x_1^2 + 4x_1x_3 + 5x_2x_3 + 3x_1 + 3x_3, 4x_2^2 + 3x_1x_3 + 4x_2x_3 + 4x_1 + 4x_3)$$

is an immersion on the 2-dimensional spheres of radius $r = 1$. In that case $\dim_{\mathbb{R}} \mathcal{A} = 20$, and $I(h|_{S^2(1)}) = 1$.

Example 29.

$$h(x_1, x_2, x_3, x_4, x_5) = (x_1x_2 + x_2, 3x_3x_5 + 2x_1, x_1^2 + x_2, x_3^2 + 3x_3, 3x_1x_5 + x_1, 4x_2x_5 + x_1, 2x_4^2 + x_4, x_3^2 + x_5)$$

is an immersion on the 4-dimensional spheres of radius $r = 1, 10$. In both cases $\dim_{\mathbb{R}} \mathcal{A} = 10$, and $I(h|_{S^4(1)}) = 0$, $I(h|_{S^4(10)}) = -1$.

Example 30.

$$h(x_1, x_2, x_3, x_4) = (x_2x_4 + x_4, 2x_1x_4 + x_3, 3x_2x_4 + 4x_1, 3x_3x_4 + x_3, x_1x_2 + x_3, 2x_2x_3 + x_1)$$

is an immersion on the 3-dimensional sphere of radius $r = 1$. Take $u = 3(x_1 - y_1) + 5(x_2 - y_2) - 2(x_4 - y_4)$. In that case $\dim_{\mathbb{R}} \mathcal{A} = 18$, and $I(h|_{S^3(1)}) \equiv 1 \mod 2$.

References

[1] Z. Szafraniec, Topological degree and quadratic forms, Journal of Pure and Applied Algebra 141 (1999) 299–314.
[2] H. Whitney, The self-intersections of a smooth n-manifold in $2n$-space, Annals of Mathematics 45 (2) (1944) 220–246.
[3] G.-M. Greuel, G. Pfister, H. Schönemann, Singular 3.0.2. A computer algebra system for polynomial computations, Centre for Computer Algebra, University of Kaiserslautern (2006) http://www.singular.uni-kl.de.
[4] D. Cox, J. Little, D. O'Shea, Ideals, Varieties and Algorithms, Second edn., 1996.
[5] D. Eisenbud, H.I. Levine, An algebraic formula for the degree of C^∞ map germ, Annals of Mathematics 106 (1977) 19–44.
[6] G.M. Khimshiashvili, On the local degree of a smooth map, Soobshch. Akad. Nauk Gruz. SSR 85 (1977) 309–311.
[7] G.M. Khimshiashvili, On the local degree of a smooth map, Trudy Tbilisi Math. Inst. 64 (1980) 105–124.
[8] E. Becker, J.P. Cardinal, M.F. Roy, Z. Szafraniec, Multivariate Bezutians, Kronecker symbol and Eisenbud-Levine formula, Progress in Mathematics 143 (1996) 79–104.
[9] E. Kunz, Kähler Differentials, Vieweg Advanced Lectures in Mathematics, Braunschweig, Wiesbaden, 1986.
[10] G. Scheja, U. Storch, Über Spurfunktionen bei vollständingen Durchschnitten, J. Reine Angew Math. 278/279 (1975) 174–190.
[11] R. Lashof, S. Smale, On the immersion of manifolds in euclidean spaces, Annals of Mathematics 68 (3) (1958) 562–583.
[12] H. Whitney, Differentiable manifolds, Annals of Mathematics 37 (3) (1936) 645–680.